SOME LOWENHEIM-SKOLEM RESULTS FOR ADMISSIBLE SETS

BY

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ABSTRACT

For any admissible set A, there is an A-recursive set of sentences of $\mathscr{L}_A$ which has a model but no A-finite model. A countable admissible set has the Lowenheim-Skolem property iff it is recurisvely inaccessible and locally countable.

$\mathscr{L}_{\infty\omega}$ is the infinitary language which allows conjunctions and disjunctions over arbitrary sets of formulas, but quantifications only over finitely many variables at a time. For an admissible set A, $\mathscr{L}_A$ means the language "$\mathscr{L}_{\infty\omega} \cap A$." An admissible set A is said to satisfy the Lowenheim-Skolem property if whenever $\mathscr{L}$ is an A-finite language and ϕ is a sentence of $\mathscr{L}_A$ which has a model, then ϕ has an A-finite model. The standard downward Lowenheim-Skolem theorem for $\mathscr{L}_{\infty\omega}$ is the statement that $H(\varkappa)$ satisfies the Lowenheim-Skolem property for each cardinal $\varkappa$. Two directions for inquiry are immediately obvious. The first direction is simply a search for a natural characterization of those admissible sets satisfying the Lowenheim-Skolem property. The second direction leads us to determine whether there is some stronger Lowenheim-Skolem property which is satisfied by some, perhaps smaller, class of admissible sets. In this paper we briefly explore both directions.

0.

We begin by recalling some notions concerning admissible sets. For the definitions, proofs or explanations that we do not include, the reader may consult [1] or [3].

If A is an admissible set, we denote by $o(A)$ the least ordinal not in A. We say

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that is recursively inaccessible iff whenever $a \in A$, there is an admissible set B with $a \in B \in A$. For a transitive set a, a^+ is the smallest admissible set containing a. Then, A is not recursively inaccessible only if there is a transitive $a \in A$ such that $o(a^+) = o(A)$. A is locally countable iff for each $a \in A$, there is a 1-1 function $f \in A$ from a into ω. A is said to be projectible iff there is an $a \in A$ and an A-recursive F mapping A 1-1 into a. We will need the following:

LEMMA. *If A is admissible and projectible into $a \in A$, then a has no power set in the sense of A.*

The proof is basically the original Cantor diagonal argument.

Rather than list the axioms for the theory KP, we will say that it is precisely that weak set theory which makes transitive $\in$-models admissible. If $\mathfrak{M} = \langle M, E \rangle$ is a model of KP, then Barwise [1] has shown that the well-founded "standard part" of M with respect to E is admissible.

1.

H. Friedman (personal communication) made the second general question specific by asking if for every admissible set A, there is an A-recursive set of sentences with a model, but no A-finite model. We answer this question in:

THEOREM 1. *Let A be admissible. Then there is an A-recursive set S of sentences of $\mathscr{L}_A$, where $\mathscr{L}$ has only ε, such that S has a model but no A-finite model.*

PROOF. First we capture the $\in$-diagram of A, using only the symbol ε, in a way suggested to us by Barwise. By $\in$-induction in A we define

$$\phi_0(v) = \forall y \neg [y \,\varepsilon\, v]$$

$$\phi_a(v) = \forall y [y \,\varepsilon\, v \leftrightarrow \bigvee_{b \in a} \phi_b(y)].$$

Let S be the A-recursive set of formulas

$$\{\exists !\, v \phi_a(v) \colon a \in A\}.$$

S obviously has $\langle A, \varepsilon \rangle$ as a model. Suppose that $\mathfrak{M} = \langle M, E \rangle$ is any model for S. Consider the mapping F from A to M defined by

$$\langle a, m \rangle \in F \leftrightarrow \mathfrak{M} \models \phi_a[m].$$

It follows that F is indeed a function since, according to S, each ϕ_a has a unique solution. F is 1-1 since A is extensional. Of course F is an isomorphic embedding of $\langle A, \varepsilon \rangle$ into $\mathfrak{M}$, such that $\mathfrak{M}$ is an end extension of the image of A. Moreover, if $\mathfrak{M}$ is A-finite, then F is A-recursive. Conversely, if $\mathfrak{M}$ is any extensional structure (full extensionality is generally not necessary) such that A can be embedded in $\mathfrak{M}$ so that $\mathfrak{M}$ is an end extension of the image of A, then $\mathfrak{M}$ is a model of S and F is the unique embedding.

We now assume that $\mathfrak{M}$ is A-finite and derive a contradiction. We know, in particular, that F is a projection of A into M, and so, by the Lemma, A cannot satisfy the power set axiom. We use our hypothesis to show that A must, in fact satisfy the power set axiom.

Let $a \in A$. Since F is an "end embedding," it is clear that for each $b \in A$,

$$b \subset a \leftrightarrow \mathfrak{M} \models F(b) \subset F(a).$$

Conversely, if $m \in M$ and $\mathfrak{M} \models m \subset F(a)$, then, since F is an "end embedding," $m \subset$ range F. Thus by Σ-replacement in A, there is a b with $F(b) = m$, and so clearly also $b \subset a$.

If we let

$$P = \{m \in M : \mathfrak{M} \models m \subset F(a)\}$$

then $P \in A$ by Δ-separation, and clearly P is the power set of $F(a)$ in $\mathfrak{M}$. Now let

$$P' = \{b : \exists m \in P[F(b) = m]\}.$$

Then $P' \in A$ by Σ-replacement, and by our observations above, P' is the power set of a in A.

This completes the proof.

Of interest in its own right is:

COROLLARY. *Let A be admissible. Let $\mathfrak{M}$ be an A-finite extensional structure. Then $\mathfrak{M}$ is not isomorphic to an end extension of $\langle A, \varepsilon \rangle$.*

2.

Barwise, in his thesis, noted that if α is the first recursively inaccessible ordinal then L_α has the Lowenheim-Skolem property. Later, Barwise, Sacks, and the present author each realized that every locally countable recursively inaccessible set satisfies the Lowenheim-Skolem property. The reason for this is that the "model existence theorem" of Makkai is provable in KP, and that "provability", or rather "non-provability" is absolute for countable admissible sets.

Our intention is to establish the converse, i.e. we would like to show.

THEOREM 2. *An admissible set $A \subset H(\omega_1)$ has the Lowenheim-Skolem property iff A is locally countable and recursively inaccessible.*

PROOF. It is easy to see that if A has the Lowenheim-Skolem property then A is locally countable. We need only make use of the language $\mathscr{L}$ with the binary relation symbol ε, and a unary function symbol f.

Let $a \in A$. Consider the sentence ϕ of $\mathscr{L}_A$ which describes the ε-diagram of the transitive closure of $\{a\} \cup \{\omega\}$, and says that f is a 1-1 function from the element having the ε-diagram of a into the element having the ε-diagram of ω. By the Lowenheim-Skolem property, ϕ has an A-finite model $\mathfrak{M} = \langle M, E, f \rangle$. By our observations in the proof of Theorem 1, we know we can map ω and a onto their respective representatives in M, inside A. Composing with f we get the required A-finite function.

Now suppose that A is not recursively inaccessible. Then there is a transitive $a \in A$ such that $o(a^+) = o(A)$.

One obvious candidate, in view of the corollary, for an A-finite sentence with a model but no A-finite model is the sentence expressing that a model with a single binary relation satisfies KP, and "has a as an element." As a contrast to the corollary we show that this candidate does not work, even if we add local countability.

We use the Barwise Compactness theorem. Let α be the first recursively inaccessible ordinal. Let $x \subset \omega$ be such that $\omega_1^x = \alpha$, and let $A = x^+ = L_\alpha(x)$. In our language $\mathscr{L}$ we have the binary relation symbol ε, and constant symbols a for each $a \in A$, and additional constant symbols M, N, and r.

Our A-recursive set of sentences $\Gamma \subset \mathscr{L}_A$ will say

I. $$\forall v[v \in a \leftrightarrow \bigvee_{b \in a} v = b]$$

for each $a \in A$, i.e. we have an end extension of A.

II. "Every ordinal is recursive in x".

This guarantees that our model will only be well-founded below α.

III. KP,

so that if we "truncate" the model at rank α we get an admissible set.

IV. "$M \models KP$"

V. "The ordinals of M have an initial segment of type β." for each $\beta \in \alpha$.

VI. $r \subset \omega \;\&\; r$ is an element of M.

VII. "$M = r^+$."

VIII. "Every element is countable."

IX. "$M \models$ "Every element is countable." "

Finally, to ensure that we trap something resembling M in the truncation we demand:

X. "M is isomorphic to N and $N \subset R(\omega + 2)$ and the isomorphism is the identity on $R(\omega + 1)$.

Now, let $\Gamma_0 \subset \Gamma$ with $\Gamma_0 \in A$. We take, as our model for Γ_0, A with each constant a interpreted by a itself. Suppose that each ordinal β mentioned in $(V) \cap \Gamma_0$ is below $\gamma > \omega + 1$. Then, we choose for M, $\langle L(\gamma^+), \varepsilon \rangle$, and since $L(\gamma^+)$ is countable in A; for N we take some copy of $L(\gamma^+)$ as required by (X). Finally, for r we take some real of $L(\gamma^+)$ coding γ, which can be found since γ is countable in $L(\gamma^+)$. Under this interpretation, we clearly have a model of Γ_0.

By the Barwise Compactness theorem we get a model $\langle B, E, \cdots \rangle$ for Γ. If A' is the "standard part" of $\langle B, E \rangle$, then A' is admissible and $o(A') = \alpha$. Furthermore if we let $\mathfrak{N}$ be the interpretation of N in B, and $\boldsymbol{r}$, the interpretation of r, then $\mathfrak{N} \in A'$, $\boldsymbol{r}$ is an element of $\mathfrak{N}$, and $\omega_1^{\boldsymbol{r}} = \alpha$. In summary, A' together with $\boldsymbol{r}$ and $\mathfrak{N}$ represent the anomaly for which we have been looking.

It is also possible to find an admissible A with $o(A) = \omega_1^{CK}$, and an A-finite ω-model of KP plus the axiom of infinity. One could prove the existence of such a pair by using the fact that if a Σ_1^1 predicate has a solution, it has a solution x of strictly lower hyperdegree than Kleene's 0, i.e., such that $\omega_1^{(x)} = \omega_1^{CK}$.

There are, nevertheless, several fruitful ways to prove Theorem 2. We choose a method which gives information about arbitrary admissible A.

We proceed as follows. We start with the original candidate. If it has no A-finite model we are done. Now suppose that it does have a model $\mathfrak{M}$.

Instead of looking at the entire model $\mathfrak{M}$, we merely look at its ordinals, which we call $\mathfrak{N}$. Clearly $\mathfrak{N}$ is A-finite.

In [2], Friedman observed that the order type of any ω-model of KP is of the form $\alpha + \alpha \cdot \rho$ where α is an admissible ordinal and ρ is dense without endpoints. In our case, of course, $\alpha = o(A)$ and ρ is simply the order type of the rationals.

Just as we showed that if $A \subseteq H(\omega_1)$ satisfies the Lowenheim-Skolem property, then A is locally countable, we could show that isomorphic A-finite structures have A-finite isomorphisms between them. We will use this property to obtain an A-finite linear ordering of type α which is, of course, impossible.

Specifically, we could describe the initial segment S of $\mathcal{N}$ of type α as the following Σ class of A.

$\{n \in \mathcal{N} : \exists\alpha\exists f[\alpha$ is an ordinal and f is an order isomorphism between $\langle\alpha, \varepsilon\rangle$ and the predecessors of n in $\mathcal{N}.\}$

At the same time we could define $\mathcal{N}\setminus S$ in the following Σ way.

$\{n \in \mathcal{N} : \exists f \exists n' \in \mathcal{N}[n \neq n'$ and f is an automorphism of $\mathcal{N}$ carrying n to $n'\}$.

This characterization works since $\mathcal{N}\setminus S$ has order type $\alpha \cdot \mathcal{N}$ and hence elements of $\mathcal{N}\setminus S$ can be non-trivially automorphed, and by our assumption automorphisms can be found within A.

Using Δ-separation in A, we have $S \in A$. Since S inherits an ordering of type α from $\mathcal{N}$, we have the desired contradiction, and so A cannot have the Lowenheim-Skolem property. This completes the proof of Theorem 2.

As a result of our use of ε-diagrams in the proof, we can observe, in addition, that the Lowenheim-Skolem property would have been no weaker had we required that the language $\mathcal{L}$ involved be strictly finite.

If A is not countable we can still get the same "negative" result, i.e., if A has the Lowenheim-Skolem property, then A is recursively inaccessible. We could adapt the above proof by considering so-called "back and forth sets" in place of automorphisms. We cannot, however, expect to get any simple "internal" conditions on A for the positive half of the result, since semantical consistency is not absolute.

REFERENCES

1. J. Barwise, *Infinitary logic and admissible sets*, J. Symbolic Logic **34** (1969), No. 2.

2. H. Friedman, *On the ordinals of models of set theory*, mimeographed notes, Stanford University.

3. H. J. Keisler, *The Model Theory of Infinitary Languages*, North Holland, Amsterdam, 1971.

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